THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MATH1010 I/J University Mathematics 2015-2016
Suggested Solution to Problem Set 3

1. (a) $\lim_{x\to 2}\dfrac{2-x}{3-\sqrt{x^2+5}} = \lim_{x\to 2}\dfrac{2-x}{3-\sqrt{x^2+5}}\cdot\dfrac{3+\sqrt{x^2+5}}{3+\sqrt{x^2+5}} = \lim_{x\to 2}\dfrac{2-x}{4-x^2} = \lim_{x\to 2}\dfrac{1}{2+x} = \dfrac{1}{4}$

(b) Let $y=\pi-x$, then when x tends to π, y tends to 0. Then
$\lim_{x\to\pi}\dfrac{\sin x}{\pi-x} = \lim_{y\to 0}\dfrac{\sin(\pi-y)}{y} = \lim_{y\to 0}\dfrac{\sin y}{y} = 1$

(c) $\lim_{x\to 0}\dfrac{\sin 6x}{\sin 5x} = \lim_{x\to 0}\dfrac{\sin 6x}{6x}\cdot\dfrac{5x}{\sin 5x}\cdot\dfrac{6}{5} = (1)(1)(\dfrac{6}{5}) = \dfrac{6}{5}$

(d) $\lim_{x\to 0}\dfrac{1-2\cos x+\cos 2x}{x^2} = \lim_{x\to 0}\dfrac{2\cos x(\cos x-1)}{x^2} = \lim_{x\to 0}\dfrac{2\cos x(-2\sin^2(\frac{x}{2}))}{x^2}$
$= \lim_{x\to 0}-\dfrac{\cos x\sin^2(\frac{x}{2})}{(\frac{x}{2})^2} = -1$

(e) $\lim_{x\to+\infty}\sqrt{x^2+x}-x = \lim_{x\to+\infty}(\sqrt{x^2+x}-x)\dfrac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x} = \lim_{x\to+\infty}\dfrac{x}{\sqrt{x^2+x}+x}$
$= \lim_{x\to+\infty}\dfrac{1}{\sqrt{1+\frac{1}{x}}+1} = \dfrac{1}{\sqrt{2}}$

(f)

$$\begin{aligned}
& \lim_{x\to+\infty} x(\sqrt{x^2+2x}-2\sqrt{x^2+x}+x) \\
= & \lim_{x\to+\infty} x(\sqrt{x^2+2x}-2\sqrt{x^2+x}+x)\left(\frac{\sqrt{x^2+2x}-2\sqrt{x^2+x}-x}{\sqrt{x^2+2x}-2\sqrt{x^2+x}-x}\right) \\
= & \lim_{x\to+\infty} x\frac{(4x^2+6x)-4\sqrt{x^4+3x^3+2x^2}}{\sqrt{x^2+2x}-2\sqrt{x^2+x}-x} \\
= & \lim_{x\to+\infty} x\left(\frac{(4x^2+6x)-4\sqrt{x^4+3x^3+2x^2}}{\sqrt{x^2+2x}-2\sqrt{x^2+x}-x}\right)\left(\frac{(4x^2+6x)+4\sqrt{x^4+3x^3+2x^2}}{(4x^2+6x)+4\sqrt{x^4+3x^3+2x^2}}\right) \\
= & \lim_{x\to+\infty} x\frac{(4x^2+6x)^2-(4\sqrt{x^4+3x^3+2x^2})^2}{\left(\sqrt{x^2+2x}-2\sqrt{x^2+x}-x\right)\left((4x^2+6x)+4\sqrt{x^4+3x^3+2x^2}\right)} \\
= & \lim_{x\to+\infty} \frac{4x^3}{\left(\sqrt{x^2+2x}-2\sqrt{x^2+x}-x\right)\left((4x^2+6x)+4\sqrt{x^4+3x^3+2x^2}\right)} \\
= & \lim_{x\to+\infty} \frac{4}{\left(\sqrt{1+\frac{2}{x}}-2\sqrt{1+\frac{1}{x}}-1\right)\left(4+\frac{6}{x}+4\sqrt{1+\frac{3}{x}+\frac{2}{x^2}}\right)} \\
= & -\frac{1}{4}
\end{aligned}$$

2. (a) $\lim_{x\to 0}\dfrac{2^x-2^{-x}}{2^x+2^{-x}} = \dfrac{1-1}{1+1} = 0$

(b) $\lim_{x\to+\infty}\dfrac{2^x-2^{-x}}{2^x+2^{-x}} = \lim_{x\to+\infty}\dfrac{1-2^{-2x}}{1+2^{-2x}} = 1$

(c) $\lim_{x\to-\infty} \frac{2^x - 2^{-x}}{2^x + 2^{-x}} = \lim_{x\to-\infty} \frac{2^{2x}-1}{2^{2x}+1} = -1$

3. Evaluate each of the following limits.

(a) $\lim_{x\to 0} \frac{\sin 2x}{x} = \lim_{x\to 0} \left(\frac{\sin 2x}{2x}\right)(2) = (1)(2) = 2$

(b) Note that $-1 \le \sin 2x \le 1$ for all $x \in \mathbb{R}$, so $-\frac{1}{x} \le \frac{\sin 2x}{x} \le \frac{1}{x}$ for all $x > 0$.
Also, $\lim_{x\to+\infty} -\frac{1}{x} = \lim_{x\to+\infty} \frac{1}{x} = 0$, by the sandwich theorem $\lim_{x\to+\infty} \frac{\sin 2x}{x}$.

4. Let $f(x) = |x+1| + |x-1|$

(a) Rewrite $f(x)$ as a piecewise defined function by filling the following blanks:

$$f(x) = \begin{cases} 2x & \text{if} \quad x \ge 1; \\ \\ 2 & \text{if} \quad -1 \le x < 1; \\ \\ -2x & \text{if} \quad x \le -1. \end{cases}$$

(b) $\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2x = 2$ and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} 2 = 2$.
Therefore, $\lim_{x\to 1+} f(x) = \lim_{x\to 1-} f(x)$ and $\lim_{x\to 1} f(x)$ exists and equals to 2.

5. Let a be a real number and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if} \quad x < 0; \\ \\ 2 & \text{if} \quad x = 0; \\ \\ a\cos x & \text{if} \quad x > 0 \end{cases}$$

Since $\lim_{x\to 0} f(x)$ exists,

$$\begin{aligned} \lim_{x\to 0^+} f(x) &= \lim_{x\to 0^-} f(x) \\ \lim_{x\to 0^+} a\cos x &= \lim_{x\to 0^-} e^{\frac{1}{x}} \\ a &= 1 \end{aligned}$$

(Remark: It is nothing related to $f(0) = 2$.)

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{n} \text{ for some } n \in \mathbb{N}; \\ \\ 0 & \qquad \text{otherwise.} \end{cases}$$

(a) Let $a_n = \frac{1}{n}$ and $b_n = \frac{\sqrt{2}}{n}$ where $n \in \mathbb{N}$.
Then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$.
However, $\lim_{n\to\infty} f(a_n) = \lim_{n\to\infty} 1 = 1$ and $\lim_{n\to\infty} f(b_n) = \lim_{n\to\infty} 0 = 0$.
Therefore, $\lim_{n\to\infty} f(a_n) \neq \lim_{n\to\infty} f(b_n)$ and $\lim_{x\to 0} f(x)$ does not exist.

(b) For $\frac{1}{4} < x < \frac{1}{2}$ and $x \neq \frac{1}{3}$, $f(x) = 0$.

Therefore, $\lim_{x \to \frac{1}{3}} f(x) = 0$.